NEWTON-RAPHSON ITERATION

Newton-Raphson iteration is a numerical technique used for finding approximations to the real roots of the equation $f(\phi)=0$ given in the form of an iterative equation

$$\phi_{n+1}=\phi_n-\frac{f(\phi_n)}{f'(\phi_n)} \tag{1}$$

where n denotes the n^{th} iteration and derivative $f'(\phi)=\frac{d}{d\phi}\left\{f(\phi)\right\}$. This iterative process can be concluded when the difference between ϕ_{n+1} and ϕ_n reaches an acceptably small value. The method is attributed to Isaac Newton (1643-1727) and Joseph Raphson (1648-1715) and some historical information is given below.

Newton-Raphson iteration can be used to solve certain types of equations that occur in surveying computations. Some examples may demonstrate its usefulness.

Example 1

In GEOM2089 Surveying 2, Assignment 2 (Subdivision Problems) there is a question (Problem Sheet 3, question 2) that eventually leads to an equation that is essentially the difference between the area of a triangle and the area of a sector of a circle. And these areas are both functions of an angle θ at the centre of a circular curve of radius R. A value for θ is required in order to solve the problem.

The equation is

$$\frac{1}{2}q\,R\sin\theta-\frac{1}{2}R^2\theta=6078.79\ \mathrm{m}^2 \tag{2}$$

Now $q=764.944$ m and the radius $R=500$ m and the equation becomes

$$191236\sin\theta-125000\,\theta=6078.79 \tag{3}$$

This is a trigonometric equation and there is no simple solution for θ but an approximation for the true value can be found by Newton-Raphson iteration.

First re-arrange equation (3) so that the right-hand side is equal to zero, giving

$$191236\sin\theta-125000\,\theta-6078.79=0 \tag{4}$$

And we may write this equation in the general form $f(\theta)=0$ where

$$f(\theta) = a\sin\theta + b\,\theta + c \tag{5}$$

and $a = 191236$, $b = -125000$, $c = -6078.79$

The derivative $f'(\theta) = \dfrac{d}{d\theta}\left\{f(\theta)\right\}$ is

$$f'(\theta) = a\cos\theta + b \tag{6}$$

And now θ can be found from the iterative formula

$$\theta_{n+1} = \theta_n - \frac{f(\theta_n)}{f'(\theta_n)} \tag{7}$$

A starting value θ_1 (the value of θ for $n=1$) must be determined before equation (7) can be used. If θ_1 is close to the true value then only a small number of iterations will be required for an acceptably accurate value. That is, the differences between results θ_k, θ_{k+1}, θ_{k+2}, ... get smaller and smaller and the process can be terminated when a difference reaches an acceptably small value. In such cases, the process is said to converge on the true value. On the other hand, if θ_1 is far from the true value, then a large number of iterations will be required for a solution, or in some cases, the process will diverge (increasing differences between successive iterations) and there will be no real solution.

In this example, we require a value of θ accurate to 1" of arc and we adopt a starting value of $\theta = 2^\circ$. The results of the iterative process are shown in Table 1.

Iteration	θ	$f(\theta)$	$f'(\theta)$	$\dfrac{f(\theta)}{f'(\theta)}$
$n=1$	2º	-3768.072979	66119.504196	-0.056989
$n=2$	5º 15' 54.79"	-16.729369	65429.096377	-0.000256
$n=3$	5º 16' 47.53"	-0.000574	65424.603105	-8.774986E-09
$n=4$	5º 16' 47.53"			

Table 1: Newton-Raphson iteration for θ

The answer $\theta = 5^\circ 16' 48''$ (nearest 1" of arc) is achieved after 3 iterations. Note that θ in equation (7) is in radians.

Example 2

In the *Geocentric Datum of Australia Technical Manual* (ICSM 2002) the formula for meridian distance m on the ellipsoid is given in the form

$$m = a\left\{B_0\phi - B_2 \sin 2\phi + B_4 \sin 4\phi - B_6 \sin 6\phi\right\} \tag{8}$$

where m is the distance along a meridian of the reference ellipsoid from the equator to the point having latitude ϕ, a is the semi-major axis of the reference ellipsoid and the coefficients B_0, B_2, B_4 and B_6 are given by

$$\begin{aligned} B_0 &= 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 \\ B_2 &= \frac{3}{8}\left(e^2 + \frac{1}{4}e^4 + \frac{15}{128}e^6\right) \\ B_4 &= \frac{15}{256}\left(e^4 + \frac{3}{4}e^6\right) \\ B_6 &= \frac{35}{3072}e^6 \end{aligned} \tag{9}$$

Equations for B_0, B_2, B_4 and B_6 are the opening terms of series expressions involving even powers of the eccentricity e of the ellipsoid. They exclude all terms greater than e^6. Note that $e^2 = f\left(2 - f\right)$ where f is the flattening of the ellipsoid.

For the Geodetic Reference System 1980 (GRS80) reference ellipsoid, where the semi-major axis a = 6378137 m and flattening f = 1/298.257222101, equation (8) can be written as

$$m = 111132.952\,549\,\phi^\circ - 16\,038.508\,412 \sin 2\phi + 16.832\,201 \sin 4\phi - 0.021\,801 \sin 6\phi \tag{10}$$

Now, suppose that the meridian distance m = 4186320 m on the GRS80 ellipsoid. What is the latitude ϕ?

There is no simple solution for ϕ, but Newton-Raphson iteration may be used to obtain an acceptable value by re-arranging equation (10) into the general form $f\left(\phi\right) = 0$ where

$$f\left(\phi\right) = A_0\,\phi^\circ - A_2 \sin 2\phi + A_4 \sin 4\phi - A_6 \sin 6\phi - m \tag{11}$$

and the derivative $f'(\phi) = \frac{d}{d\phi}\{f(\phi)\}$ is

$$f'(\phi) = A_0 - 2A_2 \cos 2\phi + 4A_4 \cos 4\phi - 6A_6 \cos 6\phi \tag{12}$$

And now ϕ can be found from the iterative formula

$$\phi_{n+1} = \phi_n - \frac{f(\phi_n)}{f'(\phi_n)} \tag{13}$$

With $A_0 = 111132.952\,549$
$A_2 = 16\,038.508\,412$
$A_4 = 16.832\,201$
$A_6 = 0.021\,801$
$m = 4185320$

The latitude ϕ is required correct to 0.0001" of arc (equivalent to approximately 0.003 m) and the results of the iterative process are shown in Table 2. A starting value for ϕ can be obtained by considering m to be an arc length on a sphere of radius a; i.e.,

$\phi = \frac{m}{a} = \frac{4186320}{6378137} = 0.656354669$ radians $= 37^\circ\, 36'\, 23''$ (nearest 1" of arc)

Iteration	ϕ	$f(\phi)$	$f'(\phi)$	$\frac{f(\phi)}{f'(\phi)}$
$n = 1$	37° 36' 23"	-22509.963230	102887.465755	-0.218782366
$n = 2$	37° 49' 30.6165"	4311.57399	103124.056405	0.041809585
$n = 3$	37° 47' 00.1020"	-2867.628390	103078.806316	-0.027819767
$n = 4$	37° 48' 40.2532"	220.161130	103108.913422	0.002135299
$n = 5$	37° 48' 32.5664"	-16.834100	103106.602360	-0.000163269
$n = 6$	37° 48' 33.1541"	1.287590	103106.779073	0.000012488
$n = 7$	37° 48' 33.1092"	-0.098480	103106.765556	-0.000000955
$n = 8$	37° 48' 33.1126"	0.00753	103106.766590	0.000000073
$n = 9$	37° 48' 33.1123"	-0.000580	103106.766511	-0.000000006
$n = 10$	37° 48' 33.1124"	0.000040	103106.766517	3.879E-10
$n = 11$	37° 48' 33.1124"			

Table 2: Newton-Raphson iteration for latitude ϕ

The answer $\phi = 37^\circ\, 48'\, 33.1124''$ (nearest 0.0001" of arc) is achieved after 10 iterations.

Some historical information on Newton-Raphson Iteration

Newton-Raphson iteration is a numerical technique used for finding approximations to the roots of real valued functions and is attributed to Isaac Newton (1643-1727) and Joseph Raphson (1648-1715). The technique evolved from investigations into methods of solving cubic and higher-order equations that were of interest to mathematicians in the 17th and 18th centuries. The great French algebraist and statesman François Viète (1540-1603) presented methods for solving equations of second, third and fourth degree. He knew the connection between the positive roots of equations and the coefficients of the different powers of the unknown quantity and it is worth noting that the word "coefficient" is actually due to Viète. Newton was familiar with Viète's work, and in portions of unpublished notebooks (circa 1664) made extensive notes on Viète's method of solving the equation $x^3 + 30x = 14356197$ and also demonstrated an iterative technique that we would now call the "secant method". In modern notation, this method for solving an equation $f\left(x\right) = 0$ is:

$$x_{n+1} = x_n - f\left(x_n\right) \Big/ \left[\frac{f\left(x_n\right) - f\left(x_{n-1}\right)}{x_n - x_{n-1}}\right]$$

In Newton's tract of 1669, *De analysi perÆquationes numero terminorum infinitus* ('On analysis by equations unlimited in the number of their terms') – chiefly noted for its initial announcement of the principle of fluxions (the calculus) – is the first recorded discussion of what we may call Newton's iterative method. He applies his method to the solution of the cubic equation $x^3 - 2x - 5 = 0$ and there is no reference to calculus in his development of the method; which suggests that Newton regarded this as a purely algebraic procedure. The process described by Newton required an initial estimate x_0 hence $x = x_0 + p$ where p is a small quantity. This was substituted into the original equation and then expanded using the binomial theorem to give a polynomial in p as

$$\begin{aligned}\left(x_0 + p\right)^3 + 2\left(x_0 + p\right) - 5 &= 0\\ x_0^3 + 3x_0^2 p + 3x_0 p^2 + p^3 - 2x_0 + 2p - 5 &= 0\\ p^3 + 3x_0 p^2 + \left(3x_0^2 + 2\right)p &= 5 - 2x_0 - x_0^3\end{aligned}$$

The second and high-order polynomial terms in p were discarded to calculate a numerical approximation p_0 from $3\left(x_0^2+2\right)p_0 = 5 - 2x_0 - x_0^3$. Now $p = p_0 + q$ (q much smaller than p_0) is substituted into the polynomial for p, giving a polynomial in q, and a numerical approximation q_0 calculated by the same manner of discarding second and higher-order terms. This laborious process was repeated until the small numerical terms, calculated at each stage, became insignificant. The final result was the initial estimate x_0 plus the results of the polynomial computations $x = x_0 + p_0 + q_0 + \cdots$ instead of successive estimates x_k being updated and then used in the next computation. This process is significantly different from the iterative technique currently used and known as Newton-Raphson.

In 1690 Joseph Raphson published *Analysis aequationum universalis* in which he presented a new method for solving polynomial equations. As an example, Raphson considers equations of the form $a^3 - ba - c = 0$ in the unknown a and proposes that if g is an estimate of the solution, then a better estimate can be obtained as $g + x$ where

$$x = \frac{c + bg - g^3}{3g^2 - b}$$

Formally, this is of the form $g + x = g - f(g)/f'(g)$ with $f(a) = a^3 - ba - c$. Raphson then applies this formula iteratively to the equation $x^3 - 2x - 5 = 0$. Raphson's formulation was a significant development of Newton's method and the iterative formulation substantially improved the computational convenience. The following comments on Raphson's technique, recorded in the Journal Book of the Royal Society are noteworthy.

"30 July 1690: Mr Halley related that Mr Ralphson [*sic*] had Invented a method of Solving all sorts of Aquations, and giving their Roots in Infinite Series, which Converge apace, and that he had desired of him an Equation of the fifth power to be proposed to him, to which he return'd Answers true to Seven Figures in much less time than it could have been effected by the Known methods of Vieta."

"17 December 1690: Mr Ralphson's Book was this day produced by E Halley, wherin he gives a Notable Improvement of ye method of Resolution of all sorts of Equations Shewing, how to Extract their Roots by a General Rule,which doubles the known figures of the Root known by each Operation, So yt by repeating 3 or 4 times he finds them true to Numbers of 8 or 10 places."

It is interesting to note here that Raphson's technique is compared to that of Viète, while Newton's method is not mentioned, although it had, by then, appeared in Wallis' *Algebra.* In the preface to his tract of 1690, Raphson refers to Newton's work but states that his own method is "not only, I believe, not of the same origin, but also, certainly, not with the same development". The two methods were long regarded by users as distinct, but the historian of mathematics, Florian Cajori writing in 1911 recommended the use of the appellation *'Newton-Raphson'* and this is now standard in mathematical texts describing Raphson's method with the notation of calculus.

The historical information above is drawn from the articles by Thomas (1990), and Tjalling (1995).

Additional historical information on this method and Thomas Simpson's contribution can be found in Kollerstrom (1992); in which the author makes a very good case for Thomas Simpson FRS (1710-61) as the inventor of the Newton-Raphson iteration. A copy of Kollerston's paper is attached as well as some pages from Thomas Simpson's paper in 1740 where he introduces his *"new Method for the Solution of all Kinds of Algebraical Equations in Numbers …";* the first application of the calculus (Newton's fluxions) to an iterative approximation technique.

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Thomas Simpson and 'Newton's method of approximation': an enduring myth

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A resurgence of interest has occurred in 'Newton's method of approximation' for deriving the roots of equations, as its repetitive and mechanical character permits ready computer use.[1] If $x = \alpha$ is an approximate root of the equation $f(x) = 0$, then the method will in most cases give a better approximation as

$$\alpha - f(\alpha)/f'(\alpha), \tag{1}$$

where $f'(x)$ is the derivative of the function into which α has been substituted.[2] Older books sometimes called it 'the Newton–Raphson method', although the method was invented essentially in the above form by Thomas Simpson, who published his account of the method in 1740.[3] However, as if through a time-warp, this invention has migrated back in time and is now matter-of-factly placed by historians in Newton's *De analysi* of 1669.[4] This paper will describe the steps of this curious historical transposition, and speculate as to its cause.

What is *today* known as 'Newton's method of approximation' has two vital characteristics: it is *iterative*, and it employs a differential expression. The latter is simply the derivative $f'(x)$ of the function, resembling a Newtonian fluxion in being based upon a theory of limits but not conceptually identical with it. The method uses the fundamental equation (1) repetitively, inserting at each stage the (hopefully) more accurate solution. This paper will argue that neither of these characteristics applies to the method of approximate solution developed by Newton in *De analysi*,[5] which also appeared in his *De methodis fluxionum et serierum infinitorum*;[6] and that the method of approximation

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1 H. Peitgen and P. Richter, *The Beauty of Fractals*, Berlin, 1986, 18.

2 See e.g. C. Tranter and C. Lambe, *Advanced Level Mathematics*, 4th edn, London, 1980, 302.

3 T. Simpson, *Essays...on Mathematics*, London, 1740, 81.

4 J. Pepper, 'Newton's mathematical work', in *Let Newton Be!* (ed. J. Fauvel *et al.*), Oxford, 1988, 63–80: 'Newton made a major breakthrough [in *De analysi*] by introducing what is now known as the *Newton–Raphson method*' (p. 73). H. Goldstine, *A History of Numerical Analysis from the Sixteenth Century through the Nineteenth*, Springer-Verlag, New York, 1977, 64–7. D. M. Burton, *The History of Mathematics, an Introduction*, 1986, 408.

5 I. Newton, *De analysi* (ed. W. Jones), London, 1711; *The Mathematical Papers of Isaac Newton* (ed. D. T. Whiteside), Cambridge, 1968, ii, 206–47. see 218–19.

6 I. Newton, *De methodis fluxionum et serierum infinitorum*', London, 1736 (English translation J. Colson), Whiteside, op. cit. (5), iii, 32–353; see 43–7, 'The reduction of affected equations'.

published by Joseph Raphson in his *Analysis aequationum universalis* of 1690[7] was iterative – indeed was the first such method to be iterative – but was not expressed in derivative or fluxional terms.

NEWTON'S METHOD

D. T. Whiteside described the method of approximate solution given in Newton's *De analysi* as 'essentially an improved version of the procedure, expounded by Viète and simplified by Oughtred'.[8] John Wallis, in the first edition of his *Algebra* (1685), extolled the method as a fine British achievement,[9] and we shall here follow his account of it. Taking the cubic equation $y^3-2y-5=0$, as the example given in *De analysi*, he started with the approximate solution of 2. Let the exact solution be $2+p$, where p is small, and substitute $2+p$ into the equation in place of y. This generates a *new* cubic equation, namely $p^3+6p^2+10p=1$. As p is small, its powers are ignored, yielding the approximate solution $10p=1$ or $p=0.1$. Next, $0.1+q$ is inserted in place of p to form another new equation, and so on. This is continued to achieve any desired level of accuracy.

That was Newton's method. To quote from W. Frend's account of it in 1796, it 'proceeds by considering the new, or transformed, equation (resulting from the substitution of $a+z$, or $a-z$, instead of x, in the original equation)'.[10] It took only the first-order terms in a binomial expansion, a subject with which Newton's *De analysi* was much concerned. It did *not* employ any fluxional calculus.

RAPHSON'S METHOD

When Raphson's method was announced to the Royal Society in July of 1690, there was emphasis on its innovative nature:

> Mr Halley related that Mr Ralphson [*sic*] had Invented a method of Solving all sorts of Aquations [...] and that he had desired of him an Equation of the fifth power to be proposed to him, to which he return'd answers true to Seven Figures in much less time than it could have been effected by the Known methods of Vieta.[11]

Raphson published his method as a tract in 1690. It had a preface referring to Newton, among several other mathematicians, in which Raphson declared that his own method was somewhat similar ('aliquid simile') to Newton's earlier account.[12] Raphson removed that preface when publishing his method as a book in 1697. He then referred solely to Viète as the ancestor of his method. Later in the work (section VI) he referred to the English mathematicians Harriot and Oughtred. An Appendix was added, referring amongst other matters to Newton's binomial theorem.

7 J. Raphson, *Analysis aequationum universalis*..., London, 1690.

8 Whiteside, op. cit. (5), ii, 218.

9 J. Wallis, *A Treatise of Algebra both Historical and Practical*, London, 1685, 338.

10 W. Frend, *The Principles of Algebra*, London, 1796, 456.

11 Journal Book of the Royal Society of London, 30 July 1690.

12 Raphson, op. cit. (7), Preface. Goldstine said of this 1690 work, 'Here Raphson acknowledges Newton as the source of the procedure' (Goldstine, op. cit. (4), 64). That is not the view here taken.

Raphson presented his method as follows. Taking as an example the equation $ba-aaa = c$ (which we would write as $a^3-ba+c=0$), let an approximate solution be g. Then, if a more accurate solution is $g+x$,

$$x=\frac{c+ggg-bg}{b-3gg}.$$

The quotient expression was obtained by a two-step procedure.[13] In the above example, one substituted $(g+x)$ for a, then expanded the power terms to give a larger equation; this was a straightforward binomial expansion. The second step was to extract the terms in x: the terms which multiplied x in this example were $(b-3gg)$, and these became the quotient.

Iterating this procedure, Raphson explained, would give any desired level of accuracy. He elaborated his method only within the context of polynomial equations, without attempting to deal with reciprocal or square root functions. His worked examples contained terms up to the seventh power.

This was Raphson's method, which he said he had derived from Viète. If it sounds odd today, it is because once the calculus technique was established, such *ad hoc* rules could be forgotten. Nowadays, it is invariably viewed in calculus terms – in the above case, the derivative of (a^3-ba+c) is $(3a^2-b)$; one divides the original function by its derivative, substituting the approximate solution g to obtain the increment x, whereby it is improved. For Raphson no such general concept appeared to be available. His book contained many pages of recipes showing how for each specific algebraic expression one could obtain the required quotient: for example the quotient for $gggg$ was $4ggg$. However, no general proofs of these recipes were provided; they were obtained using the two-step procedure.

Even after De l'Hôpital's *Analyse d'infiniments petits* was published in 1696, and rapidly became the textbook on the new Leibnizian differential and integral calculus, Raphson republished his method without alteration. As testimony to the level of British awareness of the new differential or fluxional procedures in the 1690s, this situation could have been referred to by Raphson in his *History of Fluxions* published in 1715, though this might well have compromised the staunchly pro-British tenor of its argument.

Newton's fluxional method, as it featured in Wallis's 1693 *Opera mathematica*,[14] would hardly have sufficed to deliver the required quotient term: an implicit differentiation method was there outlined which left the time-based fluxions $\dot{x}$ and $\dot{y}$ embedded in the equation.[15] To obtain the gradient of a curve it was necessary further to divide $\dot{y}$ by $\dot{x}$, a procedure which only developed in the next century. In 1695, de Moivre was still using the terms 'fluxion' and 'moment' to represent infinitely small quantities.[16] Raphson over this period evidently saw nothing to make him recast his method into a fluxional format for the second edition of his *History of Fluxions*. In this he was not alone: Halley in 1694 took

13 Raphson, op. cit. (7), 1–2.

14 J. Wallis, *Opera Mathematica*, ii, London, 1693, 391–6.

15 For a different view see H. Bos, 'Newton, Leibniz and the Leibnizian tradition' in *From Calculus to Set Theory 1630–1910* (ed. I. Grattan-Guinness), London, 1980, Ch. 2, 49–93, on 88.

16 A. de Moivre, 'Doctrinae fluxionum...', *Philosophical Transactions* (1695), 19, 52–7. See F. Cajori, *Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse*, 1919, Chicago and London, 39.

twelve pages of the *Philosophical Transactions* giving his method of solving polynomial equations by successive approximation,[17] based largely on the method of a Frenchman, Thomas de Lagny,[18] and one there finds no sign of the new fluxional method. This should not surprise us, as new ideas take a while to become accepted.

Raphson himself first referred to the fluxional method in his *Mathematical Dictionary* of 1702.[19] The reference he cited for it was Book II of Wallis's *Opera* (1693). His account was polemical, relating to the storm of controversy then gathering: Newton's fluxional method, Raphson wrote, 'passes there [Germany] and in France, under the name of Leibniz's differential calculus'. This reference made no allusion to methods of approximate solution for equations.

The Appendix to the second edition of Raphson's *Analysis* (1697) referred to the work of several contemporary mathematicians: Halley, de Lagny, Abraham Sharp (who had found π to fifty places) and then fourthly Isaac Newton. This is the sole reference to Newton in Raphson's final statement of his method, so let us be clear as to what is there acknowledged. It referred to chapter 91 of Wallis's *Algebra* of 1685, where Newton's procedures for binomial expansion were described (with infinite series for reciprocal functions), and not to chapter 94, which gave Newton's method of approximation. We may assume that he saw no need to refer to this latter section.

Raphson was there impressed by the new nomenclature for powers of variables that Newton was using, as advocated by Wallis, for example writing *aaa* as a^3. Raphson gave several examples of how his computations could be rewritten in this manner. His Appendix also referred to Newton's method of infinite series expansions; however, as none of Raphson's worked examples dealt with fractional or negative powers – which require those expansions – it is doubtful whether he can be said to have incorporated these into his method. It is quite evident that Raphson in his second edition of 1697 did not make any acknowledgement to Newton of the kind that subsequent historians have either alleged that he did, or assumed that he should have done.

SIMPSON'S METHOD

Thomas Simpson, FRS (1710–61), was a well-known British interpolationist, author of 'Simpson's rule' for obtaining the area under a curve and other results. Writing in 1740 he described 'A new Method for the Solution of Equations', making no reference to any predecessors, and affirming that: 'as it is more general than any hitherto given, it cannot but be of considerable use'.[20] It was indeed. His fine opening words were 'Take the Fluxion of the given Equation...', from which he proceeded to a version of the rule as presented in (1), using fluxions. His instructions here were: '... and having divided the whole by $\dot{x}$, let the Quotient be represented by A'. Fluxions were taken in the manner that Newton described to Wallis in 1692, which left $\dot{x}$ and $\dot{y}$ terms on each side of the equation. Dividing

17 E. Halley, 'Methodus nova accurata et facilis inveniendi radices aequationum...', *Philosophical Transactions* (1694), 18, 136–48.

18 T. F. de Lagny, *Méthodes nouvelles et abrégeés pour l'extraction et l'approximation des racines*, Paris, 1734.

19 J. Raphson, *A Mathematical Dictionary*, London, 1702.

20 T. Simpson, *Essays... on Mathematics*, 1740, Preface, vii.

through by $\dot{x}$ left what would nowadays be called the derivative of the function on the right-hand side, and dy/dx on the left. This differential expression was what Simpson referred to as 'A'. In this manner, he applied fluxions to the approximation method.

Simpson cited five examples, including a cubic equation, a square root function, a reciprocal and an exponential function.[21] It is evident that Simpson had a general command of the fluxional technique, whereby he could obtain the quotient term for the approximation formula.

Some might object, Simpson commented, that the method of fluxions 'being a more exalted Branch of the Mathematics, cannot be so properly applied to what belongs to common algebra'.[22] This indicates that he believed he was being innovative in applying the method of fluxions to this area of mathematics. His use of a fluxion in this manner sufficiently resembles the modern formulation for him to be credited (I suggest) as inventor of the method.

THE METHODS COMPARED

In the eighteenth century there was debate over whether the Newton or the Raphson method was preferable. The 'Observations on Mr Raphson's method' (1796) by W. Frend compared their relative merits, and concluded that:

> with respect to the simplicity and conception of the two methods, Mr Raphson's method seems to be preferable to Sir Isaac Newton's; because the former always refers back to the original equation $x^3-2x=5$, whereas the latter method refers to the preceding transformed equation $10z+6z^3=1$, which has more terms and larger coefficients than the original equation...I consider Mr Raphson's method of resolving them [equations] as, upon the whole, more convenient than that of Sir Isaac Newton.[23]

This view was echoed in virtually identical terms by Francis Maseres, a Fellow of the Royal Society, in a tract of nine pages comparing the two methods.[24]

J. L. Lagrange's influential treatise, *Résolution des équations numériques* of 1798, discussed the two methods. It referred to Newton's method of approximation as being well known, and refined and generalized the Newtonian method of *De analysi*, though without reference to fluxions or differentials. Lagrange expressed surprise that Raphson had not referred to Newton's earlier work, taking the view that 'ces deux méthodes ne sont au fond que le même présentée différemment', though conceding that Raphson's method was 'plus simple que celle de Newton', because 'on peut se dispenser de faire continuellement de nouvelles transformées'.[25]

These remarks we find in one of the notes at the end of Lagrange's treatise. Its main text was composed in 1767–68, and twelve notes were added in the 1790s. These notes used the '$f'(x)$' notation for the 'fonction dérivée'. He had introduced it as part of his algebraic

21 Ibid., 83–6.

22 Ibid., vii.

23 W. Frend, op. cit. (10), 456, 492.

24 F. Maseres, 'On Mr Raphson's Method of Resolving Affected Equations by Approximation', in *Bernoulli's Mathematical Tracts*, 1795, published by F. Maseres, London, 577–86, on 585.

25 J. L. Lagrange, Note V, 'Sur la méthode d'approximation donnée par Newton', in *Traité de la résolution des équations numériques*, 1st edn, Paris, 1798; 2nd edn, 1808, reprinted 1826, 122.

foundation of the calculus, based upon expanding a function $f(x+h)$ in powers of h and *defining* these 'fonctions' from the coefficients of the powers. This was his own version of the differential calculus to replace both Newton's and Leibniz';[26] yet it is striking that he did *not* use it in his note 5 comparing the approximation methods – even though he did apply it freely in various other of these notes.[27] As Raphson had done a century earlier, Lagrange treated the approximation methods solely in algebraic terms. This may remind us how very innovative Simpson was being, in applying the fluxional technique within this algebraic context.

THE MYTH BLOSSOMS

In the early nineteenth century, the mathematician Joseph Fourier presented the method in terms of the now-universal $f'(x)$ calculus notation, describing it as 'le méthode newtonienne'.[28] Fourier's writings on equations became very well known. The British mathematicians Burnside and Panton referred to the method, using the language of calculus, as being that of Newton and Lagrange, without mentioning Raphson. They did refer to Simpson and the Bernoullis as having 'occupied themselves' with the problem.[29] Similarly in Germany, Runge gave the method in Leibnizian form, attributing it to Newton.[30] Moritz Cantor reviewed the approximation methods of Newton, Raphson, Halley and de Lagny, describing Raphson as 'an absolute admirer and imitator of Newton', whose approximation method 'greatly resembled that of Newton'.[31]

Reviewing the situation in 1911, Florian Cajori concluded that the method ought properly to be called the 'Newton–Raphson method';[32] however, no person in the seventeenth or eighteenth centuries adopted such a view. Cajori's grounds for referring to the method as 'the Newton–Raphson method' may have been his view that 'If r is the approximation already reached, then Newton uses a divisor which in our modern notation takes the form $f'(r)$'.[33] However, the Newtonian method does not inherently employ a divisor, let alone one equivalent to $f'(r)$.

A recent appreciation of Joseph Raphson discussed the historically perceived difference between the two methods, concluding: 'it is actually Raphson's simpler (and therefore superior) method, not Newton's, that lurks inside millions of modern computer programs'.[34] In support of this argument it presented the familiar claim that Raphson's

26 Grattan-Guinness, op. cit. (15), Ch. 3, I. Grattan-Guinness, 'The emergence of mathematical analysis and its foundational progress, 1780–1880', p. 115.

27 Lagrange, op. cit. (25), 130–52.

28 J. B. J. Fourier, *Analyse des équations déterminées*, Paris, 1831, 169, 173 and 177.

29 W. S. Burnside and A. W. Panton, *The Theory of Equations*, London, 1881, Note B, 384–6.

30 C. Runge, 'Separation und Approximation der Wurzeln', *Encyk. der Math. Wissenschaften*, 1900, 1, 404–48, article IB3a (pp. 433–5).

31 M. Cantor, *Geschichte der Mathematik*, Leipzig, 1898, iii, 114–15; also 2nd edn (1901), 119–20.

32 F. Cajori, 'Historical note on the Newton–Raphson method of approximation', *American Mathematical Monthly* (1911), 18, 29–32, on 30.

33 Ibid., 31.

34 D. J. Thomas, 'Joseph Raphson, F.R.S.', *Notes Rec. Roy. Soc. London* (1990), 44, 151–67, on 155. Thomas here mistakenly claims (p. 155) that Newton 'never published his version' [of approximation method], but see note 6 above. In addition the text of *De analysi* was reprinted in the *Commercium epistolicum* of 1713.

method involved 'calculation of the first derivative', quoting the differential-based equation given at the start of this article. The historical record hardly supports such a viewpoint. The method of approximation inside computer programs is surely that of Simpson.

Such attitudes endure to this day, to be found even in histories of mathematics. Boyer's *History of Mathematics* (1968) affirmed that 'Newton's Method' for the approximate solution of equations could be found in *De analysi*,[35] citing its modern formulation in terms of a derivative $f'(x)$. A marginally more accurate version has appeared in *Makers of Mathematics* by Stuart Hollingdale (1989), which correctly described the method of approximation given in *De analysi*, but then blithely asserted, 'Newton also devised an iterative method... first published in its original form by Joseph Raphson in 1690'.[36]

THE LURE OF MYTH

The seeds of lasting confusion were sown by Wallis in his *Opera* of 1693:[37] he received in August 1692 historic letters from Newton, now lost,[38] containing the recipe for what would nowadays be called implicit differentiation in fluxional terms of an equation, using the newly-invented dot notation;[39] he published that method without acknowledging a contemporary source, and alleged that the method was present in Newton's letters of the 1670s sent to Leibniz, which was scarcely the case.[40] That act needs to be seen within the context of the controversy then beginning over the genesis of the new calculus methods. To quote Whiteside, 'The letters to Wallis in 1692... [were] the first significant announcement to the world at large of the power of Newton's fluxional method'.[41]

Modern scholarship has located the fairly limited extent to which Newton did compose differential equations, in the early 1690s.[42] These were reformulations of dynamical issues from his *Principia*, and did not include methods of approximate solution of equations. It seems that only at the tercentenary of these events can myth and fact be disentangled. Disputes over the birth of calculus have led mathematicians to locate such achievements at a too-early period. The myth we have surveyed is a legacy from that dispute.

Taking the time-honoured view that Raphson used differentials or fluxions in his method, where was he supposed to have got them from? This always remained unspecified. Prior to Wallis's 1693 publication, it is not evident that there was a published source from which British mathematicians could have derived such a method, had they so wished.

35 C. Boyer, *A History of Mathematics*, Princeton, 1968, reprinted 1980, 449.

36 S. Hollingdale, *Makers of Mathematics*, London, 1989, 179.

37 J. Wallis, op. cit. (14), 390.

38 *The Correspondence of Isaac Newton*, London, 1961, iii, 222–8.

39 D. T. Whiteside, 'The mathematical principles underlying Newton's *Principia*', *Journal for the History of Astronomy* (1790), 1, 116–38, on 119.

40 A. R. Hall, *Philosophers at War*, Cambridge, 1980, 94–6.

41 D. T. Whiteside, 'Essay review of *The Correspondence of Isaac Newton*, Vol. III', *History of Science* (1962), 1, 97.

42 Whiteside, op. cit. (39), 119.

It was, we have here argued, unequivocally the method of approximation invented by Thomas Simpson that Fourier restated using a derivative notation, and which has somehow come to gravitate within a Newtonian orbit. I found no source which credited Simpson as being an inventor of the method. None the less, one is driven to conclude that neither Raphson, Halley nor anyone else prior to Simpson applied fluxions to an iterative approximation technique.

This page and the following pages contain extracts from a publication by Thomas Simpson in 1740. They are the Title page, the Preface (pp. v to viii), page 1, and pages 81 to 86.
In p. vii of the Preface, Simpson describes the sixth part of his publication which is his *"new Method for the Solution of all Kinds of Algebraical Equations in Numbers ..."* This new method is what is now known as Newton-Raphson iteration.
This sixth part is contained in pages 81 to 86 where Simpson presents two cases and five examples.

This work by Simpson is referred to by Nick Kollerstrom (Thomas Simpson and 'Newton's method of approximation': an enduring myth, British Journal for the History of Science (BJHS), Vol. 25, pp. 347-54, September 1992) as evidence for his argument that Simpson should be credited with the discovery of the Newton-Raphson iteration.

ESSAYS

ON SEVERAL

Curious and Uſeful SUBJECTS,

In SPECULATIVE and MIX'D

MATHEMATICKS.

Illuſtrated by a Variety of EXAMPLES.

By *THOMAS SIMPSON.*

LONDON:

Printed by H. WOODFALL, *jun.* for J. NOURSE, at the *Lamb* without *Temple-Bar.*

M.DCC.XL.

PREFACE.

THE Reader, I preſume, will excuſe me, if, inſtead of acquainting him, in the uſual Way, with the many weighty Reaſons that induced me to publiſh the following Sheets, I ſhall take up no more of his Time than to give a conciſe Account of the Nature and Uſefulneſs of the ſeveral Papers that compoſe this Miſcellany, *in the Order they are printed.*

The firſt, then, is concerned in determining the Apparent Place of the Stars ariſing from the progreſſive Motion of Light, and of the Earth in its Orbit; which, though it be a Matter of great Importance in Aſtronomy, and allowed one of the fineſt Diſcoveries, yet had it not been fully and demonſtratively treated of by any Author, or indeed thrown into any Method of Practice. Now, however, I muſt not omit to acknowledge, that in the laſt Volume of the Memoirs *of the* ROYAL ACADEMY *of* SCIENCES, *for the Year* 1737.

a *lately*

lately publiſhed at Paris, *and brought hither a few Weeks ſince, there is a Paper on this Subject by Monſieur* Clairaut, *a very eminent Mathematician of that* Academy; *to which he ſubjoins a Set of Practical Rules for the* Aberration *in* Right-Aſcenſion *and* Declination *only; wherein moſt of his Analogies are exactly the ſame as thoſe inſerted in this Book, with which Dr.* Bevis *favoured me: For which Reaſon I think it proper to aſſure my Readers, that my Paper, together with the Doctor's Rules, were quite printed off, and in the Hands of ſeveral Friends, who deſired them, before* Chriſtmas 1739. *when the Severity of the Seaſon interrupted for a conſiderable Time the Impreſſion of this Treatiſe.*

The ſecond Paper, treats of the Motion of Bodies affected by Projectile and Centripetal Forces; wherein the Invention of Orbits and the Motion of Apſides, with many others of the moſt conſiderable Matters in the Firſt Book of Sir Iſaac Newton's PRINCIPIA, *are fully and clearly inveſtigated.*

The Third, ſhews how, from the Mean Anomaly of a Planet given, to find its true Place in its Orbit, by three ſeveral Methods; but what may beſt recommend this Paper, is the Practical Rule in the annexed Scholium, which will, I hope, be found of Service.

The Fourth, includes the Motion and Paths of Projectiles in reſiſting Mediums, in which not only the Equation of the Curve deſcribed according to any Law of Denſity, Reſiſtance, &c. but all the moſt important Matters, upon this Head, in the Second Book of the above-named illuſtrious Author, are determined in a new, eaſy, and comprehenſive Manner.

The Fifth, conſiders the Reſiſtances, Velocities, and Times of Vibration, of pendulous Bodies in Mediums.

The

The Sixth, contains a new Method for the Solution of all Kinds of Algebraical Equations in Numbers; which, as it is more general than any hitherto given, cannot but be of considerable Use, though it perhaps may be objected, that the Method of Fluxions, whereon it is founded, being a more exalted Branch of the Mathematicks, cannot be so properly applied to what belongs to common Algebra.

The Seventh, relates to the Method of Increments; which is illustrated by some familiar and useful Examples.

The Eighth, is a short Investigation of a Theorem for finding the Sum of a Series of Quantities by Means of their Differences.

The Ninth, exhibits an easy and general Way of Investigating the Sum of a recurring Series.

These three last Papers relate chiefly to the Inventions of Others: As they are all of Importance, and are required in other Parts of the Book, I could not well leave them entirely untouch'd; and if I shall be thought to have thrown any new Light upon them, that may benefit young Proficients, I have my End.

The Tenth, comprehends a new and general Method for finding the Sum of any Series of Powers whose Roots are in Arithmetical Progression, which may be applied with equal Advantage to Series of other Kinds.

The Eleventh, is concerned about Angular Sections and some remarkable Properties of the Circle.

The

The Twelfth, includes an easy and expeditious Method of Reducing a Compound Fraction to Simple Ones; the first Hints whereof I freely acknowledge to have received from Mr. Muller's *ingenious Treatise on Conic Sections and Fluxions.*

The Thirteenth and last, containing a general Quadrature of Hyperbolical Curves, is a Problem remarkable enough, as well on account of its Difficulty, as its having exercised the Skill of several great Mathematicians; but as none of the Solutions hitherto published, tho' some of them are very elegant ones, extend farther than to particular Cases, except that given in Phil. Transf. N°. 417. *without Demonstration, I flatter myself that this which I have now offered, may claim an Acceptance, since it is clearly investigated by two different Methods, without referring to what hath been done by Others, and the general Construction rendered abundantly more simple and fit for Practice than it there is.*

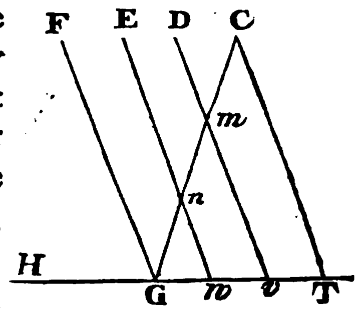
ESSAYS

On ſeveral Curious and Uſeful Subjects in Speculative *and* Mixt Mathematicks.

Of the Apparent Places of the FIXED STARS, ariſing from the *Motion of Light*, and the *Motion of the Earth in its Orbit.*

PROPOSITION I.

If the Velocity of the Earth in its Orbit bears any ſenſible Proportion to the Velocity of Light, every Star in the Heavens muſt appear diſtant from its true Place; and that by ſo much the more, as the Ratio of thoſe Velocities approaches nearer to that of Equality.



FOR, if while the Line CG is deſcribed by a Particle of Light coming from a Star in that Direction, the Eye of an Obſerver at T be carry'd, by the Earth's Motion, thro' TG; and CT be a Tube made uſe of in obſerving; and a Particle of Light, from the ſaid Star, be

B juſt

that is, if two Arcs, A, B, be taken in the ſame Ratio, as two other Arcs, C, D, the Number of Vibrations betwixt deſcribing the two former, will be to the Number betwixt deſcribing the two latter, in one whole Deſcent of the Pendulum, as C^{n-1} to A^{n-1}, or as D^{n-1} to B^{n-1}. From whence, and the foregoing Concluſions, not only the Law, but the abſolute Reſiſtance of Mediums may be found, by obſerving the Number of Vibrations performed therein by given Pendulums, in loſing given Parts of their Motion.

A new Method for the Solution of Equations in Numbers.

CASE I.

When only one Equation is given, and one Quantity (x) to be determined.

TAKE the Fluxion of the given Equation (be it what it will) ſuppoſing, x, the unknown, to be the variable Quantity; and having divided the whole by $\dot{x}$, let the Quotient be repreſented by A. Eſtimate the Value of x pretty near the Truth, ſubſtituting the ſame in the Equation, as alſo in the Value of A, and let the Error, or reſulting Number in the former, be divided by this numerical Value of A, and the Quotient be ſubtracted from the ſaid former Value of x; and from thence will ariſe a new Value of that Quantity much nearer to the Truth than the former, wherewith proceeding as before, another new Value may be had, and ſo another, *&c.* 'till we arrive to any Degree of Accuracy deſired.

Y CASE

CASE II.

When there are two Equations given, and as many Quantities (x and y) to be determined.

TAKE the Fluxions of both the Equations, conſidering x and y as variable, and in the former collect all the Terms, affected with $\dot{x}$, under their proper Signs, and having divided by $\dot{x}$, put the Quotient $= A$; and let the remaining Terms, divided by $\dot{y}$, be repreſented by B: In like manner, having divided the Terms in the latter, affected with $\dot{x}$, by $\dot{x}$, let the Quotient be put $= a$, and the reſt, divided by $\dot{y}$, $= b$. Aſſume the Values of x and y pretty near the Truth, and ſubſtitute in both the Equations, marking the Error in each, and let theſe Errors, whether poſitive or negative, be ſignified by R and r reſpectively: Subſtitute likewiſe in the Values of A, B, a, b, and let $\frac{Br - bR}{Ab - aB}$ and $\frac{aR - Ar}{Ab - aB}$ be converted into Numbers, and reſpectively added to the former Values of x and y; and thereby new Values of thoſe Quantities will be obtained; from whence, by repeating the Operation, the true Values may be approximated *ad libitum*.

Note, 1. That every Equation is firſt to be ſo reduced by Tranſpoſition, that the Whole may be equal to Nothing.

2. That, if after the firſt Operation, the Value of x or y be not found to come out pretty nearly as aſſumed, ſuch Value is not to be depended on, but a new Eſtimation made, and the Operation begun again.

3.

3. That, the above Method, for the general part, when x and y are near the Truth, doubles the Number of Places at each Operation, and only converges ſlowly, when the Diviſor A, $Ab - aB$, at the ſame time converges to nothing.

EXAMPLE I.

LET $300x - x^3 - 1000$ be given $= 0$; to find a Value of x. From $300\dot{x} - 3x^2\dot{x}$, the Fluxion of the given Equation, having expunged $\dot{x}$, (*Caſe* I.) there will be $300 - 3xx = A$: And, becauſe it appears by Inſpection, that the Quantity $300x - x^3$, when x is $= 3$, will be leſs, and when $x = 4$, greater than 1000, I eſtimate x at 3.5, and ſubſtitute inſtead thereof, both in the Equation and in the Value of A, finding the Error in the former $= 7.125$, and the Value of the latter $= 263.25$: Wherefore, by taking $\frac{7.125}{263.25} = .027$ from 3.5 there will remain 3.473 for a new Value of x; with which proceeding as before, the next Error, and the next Value of A, will come out .00962518, and 263.815 reſpectively; and from thence the third Value of $x = 3.47296351$; which is true, at leaſt, to 7 or 8 Places.

EXAMPLE II.

LET $\sqrt{1-x} + \sqrt{1-2xx} + \sqrt{1-3x^3} - 2 = 0$. This in Fluxions will be $\frac{-\dot{x}}{2\sqrt{1-x}} - \frac{2x\dot{x}}{\sqrt{1-2xx}} - \frac{9x^2\dot{x}}{2\sqrt{1-3x^3}}$, and therefore A, here, $= -\frac{1}{2\sqrt{1-x}} - \frac{2x}{\sqrt{1-2xx}} - \frac{9x^2}{2\sqrt{1-3x^3}}$; wherefore if x be ſuppoſed $= .5$, it will become -3.545: And, by ſubſtituting 0.5 inſtead of x in the given

(84)

given Equation, the Error will be found .204; therefore $\frac{.204}{-3.545}$ (equal $-$ .057) ſubtracted from .5, gives .557 for the next Value of x; from whence, by proceeding as before, the next following will be found .5516, &c.

EXAMPLE III.

LET there be given the Equations $y + \sqrt{y^2 - x^2} - 10 = 0$, and $x + \sqrt{yy + x} - 12 = 0$; to find x and y.

The Fluxions here being $\dot{y} + \frac{y\dot{y} - x\dot{x}}{\sqrt{yy - xx}}$ and $\dot{x} + \frac{y\dot{y} + \frac{1}{2}\dot{x}}{\sqrt{yy + x}}$ or $\dot{y} + \frac{y\dot{y}}{\sqrt{yy - xx}} - \frac{x\dot{x}}{\sqrt{yy - xx}}$, and $\dot{x} + \frac{\frac{1}{2}\dot{x}}{\sqrt{yy + x}} + \frac{y\dot{y}}{\sqrt{yy + x}}$, we have A equal $- \frac{x}{\sqrt{yy - xx}}$, B equal $1 + \frac{y}{\sqrt{yy + xx}}$, $a = 1 + \frac{\frac{1}{2}}{\sqrt{yy + x}}$, and $b = \frac{y}{\sqrt{yy + x}}$ (*Caſe* II.)

Let x be ſuppoſed equal 5, and y equal 6; then will R equal $-$ .68, r equal $-$ .6, A equal $-$ 1.5, B equal 2.8, a equal 1.1, b equal 9; therefore $\frac{Br - bR}{Ab - aB} = .23$, and $\frac{aR - Ar}{Ab - aB} = .37$, and the new Values of x and y equal to 5.23, and 6.37 reſpectively; which are as near the Truth as can be exhibited in three Places only, the next Values coming out 5.23263 and 6.36898.

Note, When Equations are given to be ſolved in this manner, it will be convenient, that they be firſt of all reduced to the moſt commodious Forms, to facilitate the Operations, whether into Fractions or Surds, or *vice verſa*: For Inſtance, the Equations in the laſt Example had been much

much eaſier ſolved, had they been firſt reduced, out of Surds, to $20y - xx - 100 = 0$, and $yy - xx + 25x - 144$ equal 0, or, by exterminating y, and working according to *Caſe* I. whereas, on the other hand, to have reduced the Equation, in the preceding Example, out of Surds (as is uſual in other Methods) would have rendered the Trouble of Solution almoſt inſuperable.

EXAMPLE. IV.

LET $49 \times \overline{x - \frac{x}{\overline{x+y}|^{2}}}^{\,2} - 25 \times \overline{1 - \frac{xx}{\overline{1+y}|^{2}}}^{\,2} = 0$, and $81 \times \overline{1 - \frac{xx}{\overline{1+y}|^{2}}}^{\,2} - 49 \times \overline{1\ \frac{x}{y} - \frac{xy}{\overline{1+x}|^{2}}}^{\,2} = 0$.

Here, taking the Fluxions of both the Equations, and proceeding according to *Caſe* II. we have A equal $49 \times \overline{1 + \frac{x-y}{\overline{x+y}|^{3}}} + \frac{50x}{\overline{1+y}|^{2}}$, $B = \frac{98x}{\overline{x+y}|^{3}} - \frac{50x^2}{\overline{1+y}|^{3}}$, $a = \frac{-162x}{\overline{1+y}|^{2}} + 49 \times \overline{\frac{y}{1+x} - \frac{1}{y} - \frac{2xy}{\overline{1+x}|^{3}}}$, and $b = \frac{162x^2}{\overline{1+y}|^{3}} + 49x \times \overline{\frac{1}{yy} + \frac{1}{\overline{1+x}|^{2}}}$

Suppoſe $x = .8$, and $y = .6$; then will be found $R = .45$, $r = 2.66$, $A = 68$, $B = 20.7$, $a = -131$, $b = 146$, and the next Values of x and y equal to $.799$ and $.582$; with which, repeating the Operation, the next following will come out $.79912$ and $.58138$, both which are true, at leaſt, to 4 Places: But, if a greater Exactneſs ſhould be deſired, let the Operation be once more repeated, and then the next Values will be true to double thoſe Places.

N. B. Altho' in ſeveral Caſes it happens, that the required Values, from the Equations themſelves, cannot be aſſumed

Z near

near the Truth without ſome Attention and Trouble; yet, from the Nature of the *Problem* from whence thoſe Equations are derived, when that is known, the Trouble may be avoided, and the Thing effected without any great Difficulty: For inſtance, tho' it is not eaſy to perceive, that y and x are about $\frac{6}{10}$ and $\frac{8}{10}$ in the laſt Example; yet, when it is known, that 1, x, and y, are the Sides of a Plain Triangle, wherein Lines, drawn to biſect each Angle and terminate in thoſe Sides, are to one another, reſpectively, as 5, 7, and 9, the Thing then appears evident upon the firſt Conſideration.

EXAMPLE V

LET $x^x + y^y - 1000 = 0$, and $x^y + y^x - 100 = 0$. Here we ſhall have $A = \overline{1 + L : x} \times x^x$, B equal $\overline{1 + L : y} \times y^y$, $a = \frac{y}{x} \times x^y + y^x L : y$, and b equal $\frac{x}{y} \times y^x + x^y L : x$. Now, it appearing from the firſt Equation, that the greateſt of the two required Quantities cannot be leſſer than 4, nor greater than 5; and from the firſt and ſecond together, that the Difference of x and y muſt be pretty large; otherwiſe $x^x + y^y$ could not be 10 times as great as $x^y + y^x$: I therefore take x (which I ſuppoſe the greater Number) equal 4.5, and y equal 2.5; and then by a Table of Logarithms, or otherwiſe, find the next Values of theſe Quantities to be 4.55 and 2.45; and the next following 4.5519, &c. and 2.4495, &c. reſpectively.

Of